

An analytical approximation of the luminosity distance in flat cosmologies with a cosmological constant

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We present an analytical approximation formula for the luminosity distance in spatially flat cosmologies with dust and a cosmological constant. Apart from the overall factor, the effect of non-zero cosmological constant in our formula is written simply in terms of a rational function. We also show the approximate formulae for the Dyer-Roeder distance (empty beam case) and the generalized angular diameter distance from redshift z_1 to z_2 , which are particularly useful in analyzing the gravitational lens effects. Our formulae are widely applicable over the range of the density parameter and the redshift with sufficiently small uncertainties. In particular, in the range of density parameter $0.3 \leq \Omega_m \leq 1$ and redshift $0.03 \leq z \leq 1000$, the relative error for the luminosity distance by our formula is always smaller than that of the recent work by Wickramasinghe and Ukwatta (2010). Hence, we hope that our formulae will be an efficient and useful tool for exploring various problems in observational cosmology.

§1. Introduction

Current cosmological observations indicate that the universe is spatially flat with a cosmological constant. In such a universe, calculations of the luminosity distance and the angular diameter distance require numerical integrations and elliptic functions.¹⁾ In order to simplify the repeated numerical integrations, Pen²⁾ has developed an efficient fitting formula. Recently, Wickramasinghe and Ukwatta³⁾ have shown another analytical method, which runs faster than that of Pen²⁾ and has smaller error variations with respect to redshift z . (See also Ref. 4).)

In this paper, we present yet another analytical approximation to calculate the luminosity distance as follows:

$$d_L(z, \Omega_m) = \frac{2c}{H_0} \frac{1+z}{\sqrt{\Omega_m}} \left\{ \Phi(x(0, \Omega_m)) - \frac{1}{\sqrt{1+z}} \Phi(x(z, \Omega_m)) \right\}, \quad (1.1)$$

$$\Phi(x) = \frac{1 + 1.320x + 0.4415x^2 + 0.02656x^3}{1 + 1.392x + 0.5121x^2 + 0.03944x^3}, \quad (1.2)$$

where c is the speed of light, H_0 is the Hubble constant, Ω_m is the density parameter of dust matter, related to the density parameter of vacuum energy Ω_Λ by $\Omega_m + \Omega_\Lambda = 1$, and

$$x(z, \Omega_m) = \frac{1 - \Omega_m}{\Omega_m} \frac{1}{(1+z)^3}. \quad (1.3)$$

Apart from the overall factor $1/\sqrt{\Omega_m}$, the effect of non-zero cosmological constant in our distance formula is written simply in terms of a rational function $\Phi(x)$.

The function $\Phi(x)$ has the following properties:

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1. $\Phi = 1$ for $x = 0$.
2. Φ is a monotonically decreasing function of x , $d\Phi/dx < 0$, and $\Phi \rightarrow 0.6735$ for $x \rightarrow \infty$.
3. Φ is a monotonically increasing function of Ω_m , $\partial\Phi/\partial\Omega_m > 0$, and $\Phi \rightarrow 1$ for $\Omega_m \rightarrow 1$.
4. Φ is a monotonically increasing function of z , $\partial\Phi/\partial z > 0$, and $\Phi \rightarrow 1$ for $z \rightarrow \infty$.
5. $0.6735 < \Phi(x(0, \Omega_m)) < \Phi(x(z, \Omega_m)) < 1$ for $0 < z$, $0 < \Omega_m < 1$.

Note that our approximate formula is explicitly shown to be exact when $\Omega_m = 1$:

$$d_L(z, 1) = \frac{2c}{H_0}(1+z) \left\{ 1 - \frac{1}{\sqrt{1+z}} \right\}. \quad (1.4)$$

§2. Approximation

The luminosity distance in flat cosmologies with a cosmological constant is given by

$$d_L(z, \Omega_m) = \frac{c}{H_0}(1+z) \int_{\frac{1}{1+z}}^1 \frac{da}{\sqrt{\Omega_m a + (1 - \Omega_m)a^4}}. \quad (2.1)$$

We define

$$F = \int_0^a \frac{\sqrt{\Omega_m} da'}{\sqrt{\Omega_m a' + (1 - \Omega_m)a'^4}}. \quad (2.2)$$

The power series expansion of F with respect to a around $a = 0$ yields

$$F = \sqrt{a} \left(2 - \frac{1}{7}x + \frac{3}{52}x^2 - \frac{5}{152}x^3 + \cdots \right), \quad (2.3)$$

where

$$x = \frac{1 - \Omega_m}{\Omega_m} a^3. \quad (2.4)$$

After expanding F up to $O(x^6)$, we can obtain the Padé approximant to the following order:

$$F = \sqrt{a} \frac{2 + b_1 x + b_2 x^2 + b_3 x^3}{1 + c_1 x + c_2 x^2 + c_3 x^3}, \quad (2.5)$$

where the numerical constants are determined as follows:

$$b_1 = \frac{4222975319}{1599088274}, \quad (2.6)$$

$$b_2 = \frac{1138125153117}{1288865148844}, \quad (2.7)$$

$$b_3 = \frac{7433983569773}{139933930445920}, \quad (2.8)$$

$$c_1 = \frac{635916643}{456882364}, \quad (2.9)$$

$$c_2 = \frac{14505955555}{28326706568}, \quad (2.10)$$

$$c_3 = \frac{44686179629}{1133068262720}. \quad (2.11)$$

Setting $a = 1/(1+z)$, we finally obtain Eqs. (1.1)–(1.3).

In order to compare our method with that of Ref. 3), we calculate the following relative error

$$\Delta E = \frac{|d_L^{\text{appr}} - d_L^{\text{num}}|}{d_L^{\text{num}}} \times 100 \text{ (per cent)}, \quad (2.12)$$

where d_L^{appr} and d_L^{num} represent the values of luminosity distances calculated by using approximate formula and numerical method, respectively.

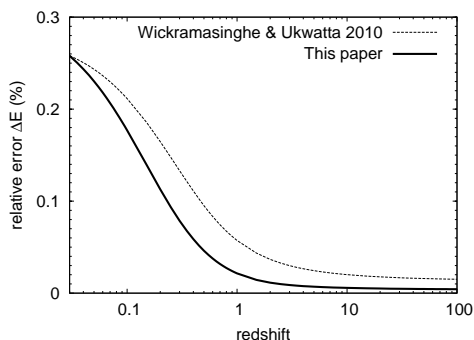


Fig. 1. A comparison of the percentage relative error ΔE for two analytical methods as a function of the redshift for $\Omega_m = 0.3$.

Fig. 1 shows a comparison of ΔE for both analytical methods for $\Omega_m = 0.3$. It is shown that our method has a smaller relative error for redshift range $0.03 \leq z \leq 100$. Although the relative error for our method is slightly worse in the range $0.01 \leq z < 0.03$, it is still less than 0.3 per cent.

Since our method is based on the Taylor expansion and the Padé approximant with respect to $x = a^3(1 - \Omega_m)/\Omega_m$, it is evident that the error in our method decreases monotonically with increasing redshift z (or increasing Ω_m), i.e., with decreasing x .

Table I. The percentage relative error $\Delta E(\%)$ for the luminosity distance by our formula Eq. (1.1)

Ω_m	$z = 0.03$	$z = 0.1$	$z = 1$	$z = 10$	$z = 1000$
0.2	1.87%	1.38%	0.25%	0.08%	0.06%
0.3	0.26%	0.18%	0.02%	0.01%	< 0.01%
0.4	0.03%	0.01%	< 0.01%	< 0.01%	< 0.01%
1	0	0	0	0	0

Table I shows the relative percentage error ΔE in our method. It is apparent that our approximate formula has sufficiently small uncertainties in the wide range of parameters. Only one exception is the nearby region (say, $z < 0.1$) in the low density (say, $\Omega_m < 0.2$) universe, where the relative error ΔE exceeds 1 per cent. In such a nearby region ($z \ll 1$), however, we may alternatively use the power series expansion around $z = 0$ with sufficient accuracy.

In particular, in the range of density parameter $0.3 \leq \Omega_m \leq 1$ and redshift $0.03 \leq z \leq 1000$, the relative error for the luminosity distance in our formula is always smaller than that of the recent work by Wickramasinghe and Ukwatta.³⁾

§3. The empty beam case

The distance formula which takes the effect of clumpy distribution of matter into account has proposed in Refs. 5), 6), and later by Dyer and Roeder⁷⁾ in more general form, which is now known as the Dyer-Roeder distance. The Dyer-Roeder luminosity distance for empty beam case in flat cosmologies is

$$D_L(z, \Omega_m) = \frac{c}{H_0} (1+z)^2 \int_{\frac{1}{1+z}}^1 \frac{a^2 da}{\sqrt{\Omega_m a + (1 - \Omega_m) a^4}}. \quad (3.1)$$

Our analytic formula for the empty beam is

$$D_L(z, \Omega_m) = \frac{2}{5} \frac{c}{H_0} \frac{(1+z)^2}{\sqrt{\Omega_m}} \left\{ \Psi(x(0, \Omega_m)) - \frac{1}{(1+z)^{\frac{5}{2}}} \Psi(x(z, \Omega_m)) \right\}, \quad (3.2)$$

$$\Psi(x) = \frac{1 + 1.256x + 0.3804x^2 + 0.0164x^3}{1 + 1.483x + 0.6072x^2 + 0.0587x^3}, \quad (3.3)$$

where x is defined in Eq. (1.3). The formula can be obtained in the same way as that in the previous section. The rational function $\Psi(x)$ has the following properties:

1. $\Psi = 1$ for $x = 0$.
2. Ψ is a monotonically decreasing function of x , $d\Psi/dx < 0$, and $\Psi \rightarrow 0.2792$ for $x \rightarrow \infty$.
3. Ψ is a monotonically increasing function of Ω_m , $\partial\Psi/\partial\Omega_m > 0$, and $\Psi \rightarrow 1$ for $\Omega_m \rightarrow 1$.
4. Ψ is a monotonically increasing function of z , $\partial\Psi/\partial z > 0$, and $\Psi \rightarrow 1$ for $z \rightarrow \infty$.
5. $0.2792 < \Psi(x(0, \Omega_m)) < \Psi(x(z, \Omega_m)) < 1$ for $0 < z, 0 < \Omega_m < 1$.

Again, our approximate formula Eq. (3.2) is exact when $\Omega_m = 1$:

$$D_L(z, 1) = \frac{2}{5} \frac{c}{H_0} (1+z)^2 \left\{ 1 - \frac{1}{(1+z)^{\frac{5}{2}}} \right\}. \quad (3.4)$$

Note that no consideration on the Dyer-Roeder distance has been taken in the previous papers, e.g., Refs. 2), 3).

Table II. The percentage relative error $\Delta E(\%)$ for the Dyer-Roeder distance (empty beam case) by our formula Eq. (3.2)

Ω_m	$z = 0.03$	$z = 0.1$	$z = 1$	$z = 10$
0.2	0.90%	0.68%	0.21%	0.15%
0.3	0.14%	0.10%	0.03%	0.02%
0.4	0.02%	0.01%	< 0.01%	< 0.01%
1	0	0	0	0

Table II shows the relative error of the our formula Eq. (3.2). The relative error for $0.3 \leq \Omega_m \leq 1$ is always less than 0.15 per cent in the range $0.03 \leq z \leq 10$. For $\Omega_m = 0.2$, the accuracy gets slightly worse, but the error is still less than 1 per cent in the same redshift range.

We omitted the error calculations for $z > 10$ for the following reasons.

First, the errors are sufficiently small in those regions, and second, the Dyer-Roeder description is relevant only in the regions where the clumpy distribution of matter becomes important.

§4. The generalized angular diameter distance

Here we present the analytic formulae for the generalized angular diameter distance from redshift $z = z_1$ to $z = z_2$, which is frequently required in analyzing the gravitational lens effects. Here we only consider the case of flat cosmologies. For the distance formulae in non-flat cases, see, e.g., Ref. 8). For the standard (filled beam) case, the angular diameter distance from redshift $z = z_1$ to $z = z_2$ is

$$d_A(z_1, z_2) = \frac{c}{H_0} \frac{1}{1+z_2} \int_{\frac{1}{1+z_2}}^{\frac{1}{1+z_1}} \frac{da}{\sqrt{\Omega_m a + (1 - \Omega_m)a^4}}. \quad (4.1)$$

Our analytical approximation is simply

$$d_A(z_1, z_2) = \frac{2c}{H_0} \frac{1}{\sqrt{\Omega_m}(1+z_2)} \left\{ \frac{1}{\sqrt{1+z_1}} \Phi(x(z_1, \Omega_m)) - \frac{1}{\sqrt{1+z_2}} \Phi(x(z_2, \Omega_m)) \right\}, \quad (4.2)$$

where $\Phi(x)$ is defined in Eq. (1.2). The relative error of our formula Eq. (4.2) for $\Omega_m = 0.3$ is less than 0.02 per cent in the range $0 \leq z_1 < 1$ for $z_2 = 1$, and less than 0.01 per cent in the range $0 \leq z_1 < 3$ for $z_2 = 3$.

For the empty beam case,

$$D_A(z_1, z_2) = \frac{c}{H_0} (1+z_1) \int_{\frac{1}{1+z_2}}^{\frac{1}{1+z_1}} \frac{a^2 da}{\sqrt{\Omega_m a + (1 - \Omega_m)a^4}}, \quad (4.3)$$

and our approximate formula is

$$D_A(z_1, z_2) = \frac{2}{5} \frac{c}{H_0} \frac{(1+z_1)}{\sqrt{\Omega_m}} \left\{ \frac{1}{(1+z_1)^{\frac{5}{2}}} \Psi(x(z_1, \Omega_m)) - \frac{1}{(1+z_2)^{\frac{5}{2}}} \Psi(x(z_2, \Omega_m)) \right\}, \quad (4.4)$$

where $\Psi(x)$ is already defined in Eq. (3.3).

The reciprocity theorem holds between the angular diameter and luminosity distances as follows: $d_L(z) = (1+z)^2 d_A(0, z)$, and $D_L(z) = (1+z)^2 D_A(0, z)$.

§5. Summary

We have presented a simple analytical approximation formula for the luminosity distance in flat cosmologies with a cosmological constant. We have also shown the approximate formulae for the Dyer-Roeder distance and the generalized angular diameter distance from redshift $z = z_1$ to $z = z_2$, which are particularly useful in analyzing the gravitational lens effects. Apart from the overall factor $1/\sqrt{\Omega_m}$, the effects of non-zero cosmological constant in our distance formulae are written simply in terms of the rational functions $\Phi(x)$ for “filled beam case” and $\Psi(x)$ for “empty beam case”. Both are monotonically decreasing functions with respect to x , and increasing ones with respect to redshift z and the density parameter Ω_m .

Our formulae are widely applicable over the range of the density parameter and the redshift with sufficiently small uncertainties. In particular, in the range

$0.3 \leq \Omega_m \leq 1$ and $0.03 \leq z \leq 1000$, the relative error for the luminosity distance by our formula is always smaller than that of the recent work by Wickramasinghe and Ukwatta.³⁾ Hence, we hope that it will be an efficient and useful tool for exploring various problems in observational cosmology, such as the statistical gravitational lensing, the cosmological parameter fitting in the magnitude-redshift relation of the supernovae, and so on.

Appendix A

— Approximations in a small redshift region —

In the region of small redshift ($z \ll 1$), the standard power series expansions can safely be used. The power series expansions of $d_L(z, \Omega_m)$ (filled beam case) and $D_L(z, \Omega_m)$ (empty beam case) around $z = 0$ are

$$d_L(z, \Omega_m) = \frac{c}{H_0} \left\{ z + \left(1 - \frac{3}{4}\Omega_m \right) z^2 + \frac{9\Omega_m - 10}{8} \Omega_m z^3 + \cdots \right\}, \quad (\text{A}\cdot 1)$$

$$D_L(z, \Omega_m) = \frac{c}{H_0} \left\{ z + \left(1 - \frac{3}{4}\Omega_m \right) z^2 + \left(\frac{9}{8}\Omega_m - 1 \right) \Omega_m z^3 + \cdots \right\}. \quad (\text{A}\cdot 2)$$

Just for reference, their Padé approximants are

$$d_L(z, \Omega_m) = \frac{c}{H_0} \frac{(12\Omega_m - 16)z + (9\Omega_m^2 + 4\Omega_m - 16)z^2}{(12\Omega_m - 16) + (18\Omega_m^2 - 20\Omega_m)z}, \quad (\text{A}\cdot 3)$$

$$D_L(z, \Omega_m) = \frac{c}{H_0} \frac{(12\Omega_m - 16)z + (9\Omega_m^2 + 8\Omega_m - 16)z^2}{(12\Omega_m - 16) + (18\Omega_m^2 - 16\Omega_m)z}. \quad (\text{A}\cdot 4)$$

Table III. The percentage relative error $\Delta E(\%)$ of the power series expansion Eq. (A·1)

Ω_m	$z = 0.01$	$z = 0.1$	$z = 0.2$	$z = 0.5$	$z = 1$
0.2	< 0.01%	< 0.01%	0.02%	0.10%	0.12%
0.5	< 0.01%	0.01%	0.09%	1.25%	8.04%
0.7	< 0.01%	0.02%	0.15%	1.80%	10.56%
1.0	< 0.01%	< 0.01%	0.05%	0.66%	3.98%

Table IV. The percentage relative error $\Delta E(\%)$ of the Padé approximant Eq. (A·3)

Ω_m	$z = 0.01$	$z = 0.1$	$z = 0.2$	$z = 0.5$	$z = 1$
0.2	< 0.01%	< 0.01%	0.05%	0.51%	2.30%
0.5	< 0.01%	< 0.01%	0.03%	0.24%	0.72%
0.7	< 0.01%	< 0.01%	< 0.01%	0.05%	0.36%
1.0	< 0.01%	< 0.01%	< 0.01%	0.09%	0.42%

The relative error of Eqs. (A·1) and (A·3) are listed in Tables III and IV. The maximal relative error in the power series expansion ΔE is 10.6 per cent at $z = 1$ in the interval $0 < z < 1$ and $0.2 \leq \Omega_m \leq 1.0$, not 37 per cent which was claimed by Pen.²⁾ The Padé approximant Eq. (A·3) shows in many cases better accuracy than the power series expansion Eq. (A·1). The relative error of the Padé approximant does not exceed 3 per cent even at redshift $z = 1$ in the range $0.2 \leq \Omega_m \leq 1.0$.

Appendix B— *An analytical approximation of the growth function* —

There is one more thing. Here we present another approximate formula which we hope to be an efficient and useful tool in observational cosmology. Heath⁹⁾ has shown that the growth function in a dust cosmology can be written as

$$D_1(a) \propto H(a) \int_0^a \frac{da'}{(a'H(a'))^3}, \quad (\text{B}\cdot 1)$$

$$D_2 \propto H(a), \quad (\text{B}\cdot 2)$$

$$H(a) = \sqrt{\Omega_m a^{-3} + (1 - \Omega_m - \Omega_\Lambda) a^{-2} + \Omega_\Lambda}. \quad (\text{B}\cdot 3)$$

Although a compact expression using the incomplete beta function has been shown in Ref. 10), so far, no analytic solution of $D_1(a)$ has been presented for $\Omega_\Lambda \neq 0$. Here we restrict ourselves to the case $\Omega_m + \Omega_\Lambda = 1$, and present an approximate formula in a simple algebraic form.

We adopt a normalization for $D_1(a)$ as

$$D_1(a) = \frac{5\Omega_m}{2} H(a) \int_0^a \frac{da'}{(a'H(a'))^3}. \quad (\text{B}\cdot 4)$$

Then, in a flat cosmology $\Omega_m + \Omega_\Lambda = 1$, our formula is

$$D_1(a) = a\sqrt{1+x} \frac{1 + 1.175x + 0.3064x^2 + 0.005355x^3}{1 + 1.857x + 1.021x^2 + 0.1530x^3}, \quad (\text{B}\cdot 5)$$

where,

$$x = \frac{1 - \Omega_m}{\Omega_m} a^3. \quad (\text{B}\cdot 6)$$

Note that our approximate formula is exact when $\Omega_m = 1$.

The well-known approximation formula for the growth function in Ref. 11), which was adopted from Ref. 12), is

$$D_1^C = \frac{5\Omega_m}{2} \frac{1}{\Omega_m^{\frac{4}{7}} - (1 - \Omega_m) + \left(1 + \frac{\Omega_m}{2}\right) \left(1 + \frac{1 - \Omega_m}{70}\right)} \quad (\text{B}\cdot 7)$$

for $\Omega_m + \Omega_\Lambda = 1$. A comparison of the relative error ΔE at $a = 1$ is shown in Table V for $\Omega_m + \Omega_\Lambda = 1$. Our formula has generally smaller relative error in the range $0.2 < \Omega_m < 1$.

Table V. A comparison of the relative error $\Delta E(\%)$ of Eq. (B.5) and Eq. (B.7) at $a = 1 (z = 0)$.

Ω_m	0.2	0.3	0.5	0.9
ΔE of Eq. (B.5)	0.19%	< 0.01%	< 0.01%	< 0.01%
ΔE of Eq. (B.7)	0.54%	0.134%	0.057%	0.019%

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